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Exactly solving a two-dimensional time-dependent coupled quantum oscillator

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Abstract. Based on the generalized linear quantum transformation theory, we present a new method to exactly solve a two-dimensional time-dependent coupled quantum oscillator, and obtain the exact formulae of the normal product form of the evolution operator, evolution matrix element, wavefunction and expectation value of an arbitrary observable.

1. Introduction

In recent years, there has been a great deal of interest in the problem of solving the multi-dimensional time-dependent quantum oscillator (TDQO) analytically [1–6], because of its extensive application in various areas of physics, such as the field of experiments with cold atoms [7, 8], the control of atoms by means of laser beams or other electric [9] and magnetic fields [10], the quantum motion of a particle in a Paul trap [11, 12] and a quantized electromagnetic field in a Fabry–Pérot cavity [13]. However, there are aspects of solving the multi-dimensional TDQO which are problematic. One difficulty is the fact that, strictly speaking, some methods [14–16] are not only complicated but it is also inconvenient to apply their results to practical problems. For example, the expectation values presented in [15] were calculated using coherent states (or particle-number states); in the Heisenberg picture, therefore, the results cannot be used to calculate the wavefunction evolution of an arbitrary initial state (or the expectation values). In addition, the system considered in [16] is single-mode TDQO, which is a special (much easier) case of the system discussed in this paper. Compared with this paper, the derivation in [16] is not only lengthy but also obscure. Also, the time evolution operator presented in [16] is expressed as the product of three operators, which is not convenient for calculation. In fact, no illustration was given in the mentioned references for calculations (e.g. solving wavefunctions, expectation values, etc) in practical problems. In this paper, we present a *real* complete analytical solution in a much more concise way by means of generalized linear quantum transformation theory (GLQT) [17–20].

GLQT universally applies to a system with constant basic operator commutator, such as $\left[\hat{x}_l, \hat{p}_k/\mathrm{i}\hbar\right] = \delta_{lk}$ ($\hat{x}_l$ and $\hat{p}_l$ are one-dimensional coordinate and momentum operators, respectively). So we have the linear unitary transformation $\hat{U}$ in two-dimensional configuration space,

$$\hat{U}\left(\frac{\hat{P}}{\mathrm{i}\hbar}, \hat{X}\right)\hat{U}^{\dagger} = \left(\frac{\hat{P}}{\mathrm{i}\hbar}, \hat{X}\right)\begin{pmatrix} A & \mathrm{i}D \\ \mathrm{i}\tilde{B} & \tilde{C} \end{pmatrix} \equiv \left(\frac{\hat{P}}{\mathrm{i}\hbar}, \hat{X}\right)M \tag{1}$$

where $\hat{P} = (\hat{p}_1, \hat{p}_2)$, $\hat{X} = (\hat{x}_1, \hat{x}_2)$, M is a 4×4 complex symplectic matrix, while A, B, C and D are 2×2 real matrices, and the following equality is the symplectic condition of M:

$$\tilde{M} \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} M = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \tag{2}$$

(I is a 2×2 unity matrix) or

$$\widetilde{BA} = BA \qquad \widetilde{CD} = CD \qquad CA + \widetilde{BD} = I \qquad AC + DB = I \quad \text{etc.}$$

$\hat{U}$ can only be determined by A, B, C, D up to an arbitrary constant factor, and $\hat{U}$ can be represented in the usual exponential, normal or anti-normal product form. In this paper, we rewrite its normal product form,

$$\hat{U} = \hat{U}^{(n)} = \frac{1}{\sqrt{\det C}} : \exp\left\{\frac{\mathrm{i}}{2}\left[\frac{1}{\hbar^2}\hat{P} D\widetilde{C^{-1}}\tilde{\hat{P}} + \frac{2}{\hbar}\hat{P}\left(1 - C^{-1}\right)\tilde{\hat{X}} + \hat{X}BC^{-1}\tilde{\hat{X}}\right]\right\} : \tag{3}$$

where the notation $: \cdots :$ represents normal product, for example, $: \hat{x}_i^2 \hat{p}_i + \hat{p}_i \hat{x}_i := \hat{p}_i \hat{x}_i^2 + \hat{p}_i \hat{x}_i$. For the case of TDQO, the equation of the evolution operator $\hat{U}$ is

$$\mathrm{i}\hbar \frac{\partial \hat{U}}{\partial t} \hat{U}^{\dagger} = \hat{H}(t) \qquad \hat{U}(0) = \hat{I} \tag{4}$$

where $\hat{U}$ is a time-dependent exponential quadratic operator. The fundamental difficulty in solving equation (4) is how to calculate $\partial \hat{U}/\partial t$, since $\hat{U}$ is a cluster function of non-commuting operators $\hat{x}_l$ and $\hat{p}_l$ $(l = 1, 2)$. Although, equation (4) has the formal solution

$$\hat{U} = \hat{T} \exp\left(-\frac{\mathrm{i}}{\hbar}\int_0^t \hat{H}(\tau)\,\mathrm{d}\tau\right) \tag{5}$$

where $\hat{T}$ is the time-ordering operator, the formal solution is not practical in actual calculations, because of $[\hat{H}(t_1), \hat{H}(t_2)] \neq 0$ for arbitrary $t_1$and t_2. In this paper, it is convenient to solve equation (4) and obtain the formulae of the evolution matrix element, wavefunction and expectation value of the observable by means of GLQT and the normal product form of the evolution operator.

2. General theory

Now, we consider the Hamiltonian for two-dimensional coupled TDQO

$$\hat{H}(t) = \frac{1}{2m}(\hat{p}_1^2 + \hat{p}_2^2) + \frac{1}{2}m\omega^2(\hat{x}_1^2 + \hat{x}_2^2) + f(\hat{p}_1\hat{x}_2 - \hat{p}_2\hat{x}_1) \tag{6}$$

where m, ω and f are time-dependent real functions.

2.1. Evolution operator

Let us take the normal product form of the evolution operator $\hat{U}$ as equation (3), then substituting equations (3) and (6) into equation (4), and utilizing equations (1) and (2), we obtain

$$\begin{aligned} &\dot{D}\tilde{A} - \dot{A}\tilde{D} = -\frac{\hbar}{m} \\ &\tilde{B}\dot{C} - \tilde{C}\dot{B} = \frac{m\omega^2}{\hbar} \\ &A\dot{C} + D\dot{B} = -f\sigma \\ &A(0) = C(0) = I \qquad B(0) = D(0) = 0 \end{aligned} \tag{7}$$

where

$$\sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \dot{A} = \frac{\mathrm{d}A}{\mathrm{d}t}.$$

Applying equation (2) to simplify equation (7), we obtain

$$\begin{aligned}
&\dot{A} - f\sigma A = \frac{\hbar}{m}\tilde{B} && A(0) = I \\
&\dot{B} + fB\sigma = -\frac{m\omega^2}{\hbar}\tilde{A} && B(0) = 0 \\
&\dot{C} + fC\sigma = \frac{m\omega^2}{\hbar}\tilde{D} && C(0) = I \\
&\dot{D} - f\sigma D = -\frac{\hbar}{m}\tilde{C} && D(0) = 0.
\end{aligned} \tag{8}$$

Considering the above equation, we choose the form of M as

$$M = \begin{pmatrix} A & \mathrm{i}D \\ \mathrm{i}\tilde{B} & \tilde{C} \end{pmatrix} = \begin{pmatrix} a\mathrm{e}^{F\sigma} & \mathrm{i}d\mathrm{e}^{F\sigma} \\ \mathrm{i}b\mathrm{e}^{-F\sigma} & c\mathrm{e}^{-F\sigma} \end{pmatrix} \tag{9}$$

where $F = \int_0^t f(\tau)\,\mathrm{d}\tau$, $ac + bd = 1$. Then equation (9) becomes

$$\begin{aligned}
&\ddot{a} + \frac{\dot{m}}{m}\dot{a} + \omega^2 a = 0 && a(0) = 1 && \dot{a}(0) = 0 \\
&\ddot{d} + \frac{\dot{m}}{m}\dot{d} + \omega^2 d = 0 && d(0) = 0 && \dot{d}(0) = -\frac{\hbar}{m(0)} \\
&\ddot{b} - \left(\frac{\dot{m}}{m} + 2\frac{\dot{\omega}}{\omega}\right)\dot{b} + \omega^2 b = 0 && b(0) = 0 && \dot{b}(0) = -\frac{m(0)\omega^2(0)}{\hbar} \\
&\ddot{c} - \left(\frac{\dot{m}}{m} + 2\frac{\dot{\omega}}{\omega}\right)\dot{c} + \omega^2 c = 0 && c(0) = 1 && \dot{c}(0) = 0.
\end{aligned} \tag{10}$$

So far, we have changed the problem of solving the evolution operator into a second-order linear ordinary differential equation. For any actual problem, once $m(t)$, $\omega(t)$ and $f(t)$ are given, we can obtain the exact solution of A, B, C, D and further determine the normal product form of the evolution operator $\hat{U}$ by equation (3).

2.2. Evolution matrix elements

From equations (3) and (9), we can find the evolution matrix element between the eigenvectors of the coordinate $|X\rangle = |x_1\rangle\,|x_2\rangle$ and momentum $|P\rangle = |p_1\rangle\,|p_2\rangle$,

$$\langle P|\hat{U}|X\rangle = \frac{1}{2\pi\hbar c}\exp\left[\frac{\mathrm{i}d}{2c\hbar^2}P\tilde{P} - \frac{\mathrm{i}}{\hbar c}(\cos F\cdot P\tilde{X} + \sin F\cdot P\sigma\tilde{X}) + \frac{\mathrm{i}b}{2c}X\tilde{X}\right] \tag{11}$$

and we can find the evolution matrix element in the coordinate representation

$$\langle X'|\hat{U}|X\rangle = \frac{1}{2\pi\mathrm{i}d}\exp\left\{\frac{1}{2\mathrm{i}d}[cX'\widetilde{X}' - 2\cos F\cdot X'\widetilde{X} + 2\sin F\cdot X'\sigma\tilde{X} + aX\tilde{X}]\right\}. \tag{12}$$

The matrix element of $\hat{U}$ between the arbitrary states $|\varphi\rangle$ and $|\psi\rangle$ is

$$\langle\varphi|\hat{U}|\psi\rangle = \frac{1}{2\pi\mathrm{i}d}\int\exp\left\{\frac{1}{2\mathrm{i}d}[cX'\widetilde{X}' - 2\cos F\cdot X'\widetilde{X} + 2\sin F\cdot X'\sigma\tilde{X} + aX\tilde{X}]\right\} \times\varphi^*\left(X'\right)\psi(X)\,\mathrm{d}X'\,\mathrm{d}X \tag{13}$$

where $\varphi^*(X') = \langle\varphi|X'\rangle$, $\psi(X) = \langle X|\psi\rangle$, $dX = dx_1\,dx_2$ and $dX' = dx_1'\,dx_2'$. Applying the above formulae of the evolution matrix element, we can calculate the transition or scattering probabilities.

2.3. Wavefunction

If the initial wavefunction of TDQO is $\psi(X,0) = \langle X|\psi(0)\rangle$, then the wavefunction at arbitrary time can be expressed as

$$\psi(X,t) = \frac{\exp((c/2\mathrm{i}d)X\tilde{X})}{2\pi\mathrm{i}d}\int \exp\left[\frac{1}{\mathrm{i}d}\left(\tfrac{1}{2}aY\tilde{Y} - \cos F\cdot X\tilde{Y} + \sin F\cdot Y\sigma\tilde{X}\right)\right]\psi(Y,0)\,dY \tag{14}$$

where $\psi(X,t) = \langle X|\hat{U}|\psi(0)\rangle$.

2.4. The expectation value

For an arbitrary observable $\hat{Q}(\hat{P},\hat{X})$, from equation (1) we obtain

$$\hat{U}^\dagger\hat{Q}(\hat{P},\hat{X})\hat{U} = \hat{Q}\left(\hat{P}C + \hbar\hat{X}B, \hat{X}\tilde{A} - \frac{1}{\hbar}\hat{P}\tilde{D}\right) \equiv \hat{G}(\hat{P},\hat{X}). \tag{15}$$

Therefore, the expectation value of $\hat{Q}$ in state $|\psi(t)\rangle$

$$\bar{Q}(t) = \langle\psi(t)|\hat{U}|\psi(t)\rangle = \langle\psi(0)|\hat{U}^\dagger\hat{Q}\hat{U}|\psi(0)\rangle = \bar{G}(0) \tag{16}$$

equals the expectation value of observable $\hat{G}$ in the initial state $|\psi(0)\rangle$.

3. Applications

If the initial wavefunction is a two-dimensional *Gaussian* wavepacket, namely

$$\psi(X,0) = \sqrt{\frac{\delta_1\delta_2}{\pi}}\exp\left[-\tfrac{1}{2}\left(\delta_1^2x_1^2 + \delta_2^2x_2^2\right)\right] \tag{17}$$

then the wavefunction can be obtained from equation (14),

$$\psi(X,t) = \lambda\exp\left\{-\tfrac{1}{2}\left[\left(\tau_1\cos^2F + \tau_2\sin^2F\right)x_1^2 + \left(\tau_1\sin^2F + \tau_2\cos^2F\right)x_2^2 + (\tau_2-\tau_1)(\sin 2F)x_1x_2\right]\right\} \tag{18}$$

where

$$\lambda = \left[\frac{\delta_1\delta_2}{\pi\left(a - \mathrm{i}\delta_1^2d\right)\left(a - \mathrm{i}\delta_2^2d\right)}\right]^{1/2}$$

and

$$\tau_l = \frac{c\delta_l^2 - \mathrm{i}b}{a - \mathrm{i}\delta_l^2d}.$$

In particular, if $\delta_1 = \delta_2 = \delta$, then

$$\psi(X,t) = \frac{\delta}{\sqrt{\pi}\left(a - \mathrm{i}\delta^2d\right)}\exp\left[-\frac{1}{2}\frac{c\delta^2 - \mathrm{i}b}{a - \mathrm{i}\delta^2d}\left(x_1^2 + x_2^2\right)\right]. \tag{19}$$

It is apparent that $\psi(X,t)$ is still a *Gaussian* wavepacket. From equation (16), we obtain

$$
\begin{aligned}
&\bar{x}_l(t) = \bar{p}(t) = 0 \qquad (l = 1, 2)\\
&\overline{x_1^2}(t) = \tfrac{1}{2}\cos^2 F\left(\frac{a^2}{\delta_1^2} + \delta_1^2 d^2\right) + \tfrac{1}{2}\sin^2 F\left(\frac{a^2}{\delta_2^2} + \delta_2^2 d^2\right)\\
&\overline{x_2^2}(t) = \tfrac{1}{2}\sin^2 F\left(\frac{a^2}{\delta_1^2} + \delta_1^2 d^2\right) + \tfrac{1}{2}\cos^2 F\left(\frac{a^2}{\delta_2^2} + \delta_2^2 d^2\right)\\
&\overline{p_1^2}(t) = \tfrac{1}{2}\hbar^2\cos^2 F\left(\frac{b^2}{\delta_1^2} + \delta_1^2 c^2\right) + \tfrac{1}{2}\hbar^2\sin^2 F\left(\frac{b^2}{\delta_2^2} + \delta_2^2 c^2\right)\\
&\overline{p_2^2}(t) = \tfrac{1}{2}\hbar^2\sin^2 F\left(\frac{b^2}{\delta_1^2} + \delta_1^2 c^2\right) + \tfrac{1}{2}\hbar^2\cos^2 F\left(\frac{b^2}{\delta_2^2} + \delta_2^2 c^2\right)
\end{aligned}
\tag{20}
$$

and

$$
\begin{aligned}
&\overline{(\triangle X)^2}(t) = \overline{x_1^2}(t) + \overline{x_2^2}(t) = \tfrac{1}{2}a^2\left(\frac{1}{\delta_1^2} + \frac{1}{\delta_2^2}\right) + \tfrac{1}{2}d^2\left(\delta_1^2 + \delta_2^2\right)\\
&\overline{(\triangle P)^2}(t) = \overline{p_1^2}(t) + \overline{p_2^2}(t) = \tfrac{1}{2}\hbar^2 b^2\left(\frac{1}{\delta_1^2} + \frac{1}{\delta_2^2}\right) + \tfrac{1}{2}\hbar^2 c^2\left(\delta_1^2 + \delta_2^2\right)\\
&\overline{(\triangle X)^2(\triangle P)^2}(t) = \overline{(\triangle X)^2(\triangle P)^2}(0) + \tfrac{1}{4}\hbar^2\left[ab\left(\frac{1}{\delta_1^2} + \frac{1}{\delta_2^2}\right) - cd(\delta_1^2 + \delta_2^2)\right]^2.
\end{aligned}
\tag{21}
$$

It is clear that $\overline{(\triangle X)^2(\triangle P)^2}(t) \geqslant \overline{(\triangle X)^2(\triangle P)^2}(0)$, and $\overline{(\triangle x_l)^2}(t)$ and $\overline{(\triangle p_l)^2}(t)$ $(l = 1, 2)$ are affected by the coupled term of $\hat{H}(t)$ (see equation (6)), while $\overline{(\triangle X)^2}(t)$ and $\overline{(\triangle P)^2}(t)$ are not.

The crux of solving TDQO is to find the solution to equation (10). Now, we consider the case with arbitrary $f(t)$ and

$$
m(t) = \frac{1}{\omega(t)}\exp\left(\tfrac{1}{2}s\Omega\right)
\tag{22}
$$

where $\Omega(t) = \int_0^t \omega(\tau)\,\mathrm{d}\tau$, and s is a real constant. Substituting equation (22) into equation (10), we have

$$
a = \mathrm{e}^{-s\Omega}\begin{cases}
\left[\cosh\left(\sqrt{s^2-1}\,\Omega\right) + \dfrac{s}{\sqrt{s^2-1}}\sinh\left(\sqrt{s^2-1}\,\Omega\right)\right] & \left(s^2 > 1\right)\\
(1 + s\Omega) & \left(s^2 = 1\right)\\
\left[\cos\left(\sqrt{1-s^2}\,\Omega\right) + \dfrac{s}{\sqrt{1-s^2}}\sin\left(\sqrt{1-s^2}\,\Omega\right)\right] & \left(s^2 < 1\right)
\end{cases}
\tag{23}
$$

$$
c = \mathrm{e}^{s\Omega}\begin{cases}
\left[\cosh\left(\sqrt{s^2-1}\,\Omega\right) - \dfrac{s}{\sqrt{s^2-1}}\sinh\left(\sqrt{s^2-1}\,\Omega\right)\right] & \left(s^2 > 1\right)\\
(1 - s\Omega) & \left(s^2 = 1\right)\\
\left[\cos\left(\sqrt{1-s^2}\,\Omega\right) - \dfrac{s}{\sqrt{1-s^2}}\sin\left(\sqrt{1-s^2}\,\Omega\right)\right] & \left(s^2 < 1\right)
\end{cases}
\tag{24}
$$

$$
b = -\frac{\mathrm{e}^{s\Omega}}{\hbar}\begin{cases}
\dfrac{1}{\sqrt{s^2-1}}\sinh\left(\sqrt{s^2-1}\,\Omega\right) & \left(s^2 > 1\right)\\
\Omega & \left(s^2 = 1\right)\\
\dfrac{1}{\sqrt{1-s^2}}\sin\left(\sqrt{1-s^2}\,\Omega\right) & \left(s^2 < 1\right)
\end{cases}
\tag{25}
$$

$$d = -\hbar e^{-s\Omega} \begin{cases} \dfrac{1}{\sqrt{s^2-1}} \sinh\left(\sqrt{s^2-1}\,\Omega\right) & \left(s^2 > 1\right) \\ \Omega & \left(s^2 = 1\right) \\ \dfrac{1}{\sqrt{1-s^2}} \sin\left(\sqrt{1-s^2}\,\Omega\right) & \left(s^2 < 1\right). \end{cases} \tag{26}$$

Substituting equations (23)–(26) into equation (21), noticing that $\overline{(\triangle X)^2}(t)$ decreases in accordance with the exponential $e^{-s\Omega}$, if s and $\omega > 0$; otherwise, if $s < 0$ and $\omega > 0$, $\overline{(\triangle P)^2}(t)$ decreases. Namely, the TDQO with $m(t)$ and $\omega(t)$ satisfying equation (22) has a squeezed effect for the initial *Gassian* wavepacket.

The method presented in this paper is applicable for multi-dimensional coupled TDQO. Utilizing the above results, we can conveniently change the quantum operation into a c-number operation. Specially, we can obtain exact numerical solutions for TDQO with those cases which have no explicit expressions.

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